Lecture 17

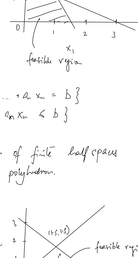
Friday, 30 October 2026 8:50 AM

Linear Programming

or Optimizing

Problem of minimizing/maximizing a linear fn of n variables, subject to linear constraints:

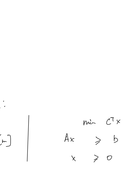
Eg: minimize $x_1 + x_2$
s.t. $x_1 + 2x_2 \leq 3$
$2x_1 + x_2 \leq 3$
$x_1, x_2 \geq 0$



hyperplane: $\{x \in \mathbb{R}^n : a_1x_1 + a_2x_2 + \dots + a_nx_n = b\}$
halfspace: $\{x \in \mathbb{R}^n : a_1x_1 + \dots + a_nx_n \leq b\}$

A polyhedron is the intersection of finite halfspaces
The feasible set of an LP is a polyhedron.

Eg minimize $x_1 + x_2$
s.t. $x_1 - x_2 \geq 0$
$-x_1 - x_2 \geq -3$



A polytope is a bounded polyhedron

An LP in general form is:

$$\min / \max_{x \in \mathbb{R}^n} \; c^T x$$

s.t.
$a_i^T x \geq b_i \quad i \in M_1$
$a_i^T x \leq b_i \quad i \in M_2$
$a_i^T x = b_i \quad i \in M_3$
$x_j \geq 0 \quad j \in N_1$
$x_j \leq 0 \quad j \in N_2$

(Some variables may be "free")

Any LP can be equivalently written as:

$$\min c^T x \quad \text{s.t.} \quad a_i^T x \geq b_i \;\; i \in [m], \quad x \geq 0$$

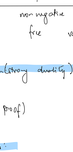
$$\min c^T x \quad Ax \geq b \quad A \in \mathbb{R}^{m \times n}, \quad x \geq 0$$

This is without loss of generality, since:

$\max c^T x \equiv \min -c^T x$
$a_i^T x \leq b_i \equiv -a_i^T x \geq -b_i$
$a_i^T x = b_i \equiv a_i^T x \geq b_i$
$-a_i^T x \geq -b_i$
$x_j \leq 0 \equiv$ replace x_j by $-y_j$, $y_j \geq 0$
x_j "free" $\equiv$ replace x_j by $x_j^+ - x_j^-$, $x_j^+, x_j^- \geq 0$

Solving LPs:
A polyhedron $P = \{x \in \mathbb{R}^n : Ax \geq b\}$
"Corners" in polyhedra are very important for LPs.

- a constraint $a_i^T x \geq b_i$ is tight / binding at x^* if $a_i^T x^* = b_i$
- constraints $a_1^T x \geq b_1, \; a_2^T x \geq b_2, \dots a_k^T x \geq b_k$ are linearly independent if $a_1, a_2, \dots a_k$ are linearly independent.
- Given P, x^* is a basic solution if n $\leq$ constraints are tight at x^*
  (x^* is the unique pt in $\mathbb{R}^n$ where the constraints are tight)
- # of basic solutions is at most $\binom{m}{n}$
- x^* is a basic feasible solution (bfs) if x^* is a basic solution & $x^* \in P$.



Theorem Suppose polyhedron P has at least one bfs. Consider the objective of minimizing $c^T x : x \in P$. Then either the optimal value is $-\infty$, or there is an optimal bfs.

Corollary If P is a polytope, there is an optimal bfs.

Thus for polytopes, we have a simple (computational then) algo for finding an optimal soln:
① Enumerate all basic solutions ($\binom{m}{n}$ basic sols)
② check if each is feasible
③ choose bfs that minimizes $c^T x$.

Duality.

max $x_1 + x_2$
$2x_1 + x_2 \leq 3 \quad : y_1$
$x_1 + 2x_2 \leq 3 \quad : y_2$
$x_1, x_2 \geq 0$

min $3y_1 + 3y_2$
$2y_1 + y_2 \geq 1$
$y_1 + 2y_2 \geq 1$
$y_1, y_2 \geq 0$

This the dual LP.

$$\min c^T x \quad Ax \geq b, \; x \geq 0 \qquad \max b^T y \quad A^T y \leq c, \; y \geq 0$$

Weak Duality Let x, y be feasible solns to the primal & dual.
Then $c^T x \geq b^T y$

Proof: $c^T \geq y^T A$. (y feasible)
$c^T x \geq y^T A x$ ($x \geq 0$)
$\geq y^T b$ ($y \geq 0$, x feasible)

$$\begin{aligned} c^T x &\geq \sum_i y_i (\sum_j A_{ij} x_j) \\ &= \sum_j x_j (\sum_i A_{ij} y_i) \\ &\geq \sum_j x_j b_j = y^T b \end{aligned}$$

∎

More general LPs:

(P) $\min c^T x$
$A_i x \geq b_i \quad i = 1 \dots d$
$A_i x = b_i \quad i = d+1 \dots m$
$x_j \geq 0 \quad j = 1 \dots k$
x_j free $\quad j = k+1 \dots n$

(D) $\max b^T y$
$(A^T)_j y \leq c_j \quad j = 1 \dots k$
$(A^T)_j y = c_j \quad j = k+1 \dots n$
$y_i \geq 0 \quad i = 1 \dots d$
y_i free $\quad i = d+1 \dots m$

(prove weak duality here yourself)

in primal / dual	in dual / primal
non-negative variable	inequality constraint
free variable	equality constraint

Theorem (strong duality): Let x^*, y^* be optimal for the primal & dual respectively. Then $c^T x^* = b^T y^*$.

(w/o proof)

Cases

Primal	Dual
Optimal value exists	Optimal value exists
Unbounded optimal	Infeasible
Infeasible	Infeasible

Can't have primal has a finite optimal, dual infeasible.

For any LP & the dual, given x^*, y^* that are feasible for the primal & dual, & $c^T x^* = b^T y^*$. Then x^*, y^* are optimal solns.

Complementary slackness: Given an LP & dual, x^*, y^* are optimal iff:
x^*, y^* are feasible, And:
∀ primal constraints: either $A_i x = b_i$ or $y_i = 0$
∀ dual constraints: either $(A^T)_j y = c_j$ or $x_j = 0$

(see extra proof of weak duality)